

Chapman-Enskog derivation of the generalized Smoluchowski equation

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Abstract

We use the Chapman-Enskog method to derive the Smoluchowski equation from the Kramers equation in a high friction limit. We consider two main extensions of this problem: we take into account a uniform rotation of the background medium and we consider a generalized class of Kramers equations associated with generalized free energy functionals. We mention applications of these results to systems with long-range interactions (self-gravitating systems, 2D vortices, bacterial populations,...). In that case, the Smoluchowski equation is non-local. In the limit of short-range interactions, it reduces to a generalized form of the Cahn-Hilliard equation. These equations are associated with an effective generalized thermodynamical formalism.

Keywords: Kinetic theory; Generalized thermodynamics; Generalized Fokker-Planck equations

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I. INTRODUCTION

The Kramers (or Klein-Kramers) equation [1, 2] and the associated Smoluchowski equation [3] are widely discussed in the physics of Brownian motion and stochastic processes. Initially, the Kramers equation was introduced to describe the statistical evolution of a gas of Brownian particles in a potential well. It has been applied in different domains of physics such as colloidal suspensions, stellar systems, chemical reactions rate theory, nuclear dynamics, just to mention a few [4]. The Kramers equation forms a particular class of Fokker-Planck equations with a constant diffusion coefficient and a linear friction [5]. In the large time limit, the velocity distribution of the particles is close to the Maxwellian distribution and the evolution of the spatial density is described by the Smoluchowski equation. The passage from Kramers to Smoluchowski equations has been the subject of many papers. In the early days, this passage was informal and qualitative [1, 4]. The derivation of the Smoluchowski equation from the Kramers equation is now a classical problem in many textbooks of statistical mechanics. In his monograph, van Kampen presents an expansion procedure similar to the Hilbert perturbation scheme [6].

In this paper, we propose another derivation of the Smoluchowski equation based on the Chapman-Enskog expansion. Usually, the Chapman-Enskog perturbation scheme is used to derive the hydrodynamical equations (Euler, Navier-Stokes,...) from the kinetic Boltzmann equation [7]. In this context, the small parameter is played by the typical collision time τ . The Chapman-Enskog expansion allows to determine the expression of the diffusion coefficients (viscosity, heat conductivity,...) in the fluid equations. In the context of the Kramers equation, the small parameter is played by the inverse of the friction coefficient ξ . We can therefore expand the distribution function in powers of ξ^{-1} . To first order in this expansion, we obtain the Smoluchowski equation. The diffusion coefficient is given by the Einstein relation $D = k_B T / \xi m$ where T is the temperature.

We shall consider two main generalizations of this problem. First, we consider the situation in which the ambient medium is uniformly rotating so that the velocity which appears in the Kramers equation is the *relative* velocity with respect to the background flow. Inertial forces appear in the rotating frame and the expression of the Smoluchowski equation is modified. Secondly, we consider a generalized class of Kramers equations introduced recently by Kaniadakis [8], Frank [9] and Chavanis [10, 11] (see also an early version in [12]). These

equations share the same properties as the ordinary Kramers equation but they are associated with a more general form of free energy functional. This generalization encompasses the case of quantum statistics (fermions and bosons) and it can also find applications in the physics of complex media. In that context, they extend the nonlinear Fokker-Planck equation [13, 14] associated with the Tsallis entropy [15]. We derive here the generalized Smoluchowski equation from the generalized Kramers equation by using a systematic procedure. The diffusion coefficient is given by a generalized Einstein relation which depends on a function $\phi(x)$ implicitly determined by a second order differential equation.

The paper is organized as follows. In Sec. II, we recall the main properties of the (generalized) Kramers and Smoluchowski equations. In Sec. III, we apply the Chapman-Enskog expansion to the generalized Kramers equation. We account for a solid rotation of the background medium and calculate the new terms that appear in the Smoluchowski equation due to rotation. In Sec. IV, we derive a variant of the generalized Smoluchowski equation by using a moment method. Finally, in Sec. V, we shortly discuss the application of these results to the case of systems with long-range interactions (self-gravitating systems, 2D vortices, bacterial populations,...). In that case, the Smoluchowski equation becomes non-local and possesses a rich mathematical and physical structure [10]. We determine the Lyapunov functional associated with the generalized Smoluchowski equation and interpret this functional as a free energy. In the limit of short-range interactions, this functional reduces to the Landau free energy and the non-local generalized Smoluchowski equation reduces to a generalized form of the Cahn-Hilliard equation. In Appendix A, we consider the case of an isotropic BGK operator and in Appendix B, we consider a wider class of Kramers equations with time varying Lagrange multipliers accounting for integral constraints (energy, angular momentum and impulse conservation).

II. GENERALIZED KRAMERS AND SMOLUCHOWSKI EQUATIONS

It has been noted recently that standard kinetic equations (Boltzmann, Landau, Kramers, Smoluchowski,...) could be generalized so that they increase a larger class of functionals than the Boltzmann entropy (or Boltzmann free energy) [11]. Such generalized kinetic equations appear when the transition probabilities have an expression different from the one we would naively expect [8] or when the system is described by a stochastic process involving

a multiplicative noise depending on the density [10, 16]. These generalized kinetic equations encompass the class of quantum kinetic equations (with exclusion or inclusion principle) already discussed in the literature. They can also be interpreted as “effective equations” trying to take into account “hidden constraints” in the physics of complex media [11].

The generalized Kramers equation can be written in the form [10]:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \cdot \left\{ D \left[f C''(f) \frac{\partial f}{\partial \mathbf{v}} + \beta f \mathbf{v} \right] \right\}, \quad (1)$$

where $f = f(\mathbf{r}, \mathbf{v}, t)$ is the distribution function, C is a convex function (i.e. $C'' > 0$), $\beta = 1/T$ is the inverse temperature and $\Phi(\mathbf{r})$ is an external potential. Since the temperature T is fixed, the above equation describes a canonical situation. We introduce the functional

$$F[f] = \frac{1}{2} \int f v^2 d^3 \mathbf{r} d^3 \mathbf{v} + \int \rho \Phi d^3 \mathbf{r} + T \int C(f) d^3 \mathbf{r} d^3 \mathbf{v}, \quad (2)$$

where $\rho = \int f d^3 \mathbf{v}$ is the spatial density. This functional can be interpreted as a free energy $F = E - TS$, where S is a “generalized entropy” and $E = K + W$ is the energy including a kinetic term and a potential term. When $S[f]$ is the Boltzmann entropy, Eq. (1) reduces to the ordinary Kramers equation. More generally, it is straightforward to check that $F[f]$ plays the role of a Lyapunov functional satisfying $\dot{F} \leq 0$. This is similar to a canonical version of the H-theorem. Finally, a stationary solution of Eq. (1) is determined by the condition $\partial f / \partial t = 0$, implying $\dot{F} = 0$, and yielding

$$C'(f_{eq}) = -\beta \left(\frac{v^2}{2} + \Phi(\mathbf{r}) \right) - \alpha. \quad (3)$$

The equilibrium distribution $f_{eq}(\mathbf{r}, \mathbf{v})$ minimizes the free energy $F[f]$ at fixed mass and temperature. Similarly, the generalized Kramers equation (1) maximizes the rate of dissipation of free energy \dot{F} at fixed mass and temperature. We refer to Ref. [10] for a more detailed discussion of these results.

For large times $t \gg \xi^{-1}$, where $\xi = D\beta$ is the friction coefficient, the distribution function is close to the *isotropic* distribution function $f(\mathbf{r}, \mathbf{v}, t)$ determined by the relation

$$C'(f) = -\beta \left[\frac{v^2}{2} + \lambda(\mathbf{r}, t) \right]. \quad (4)$$

This distribution function cancels out the diffusion current in the generalized Kramers equation (1). The pressure $p = \frac{1}{3} \int f v^2 d^3 \mathbf{v} = p(\lambda)$ and the density $\rho = \int f d^3 \mathbf{v} = \rho(\lambda)$ are related to each other by a barotropic equation of state $p = p(\rho)$ which is entirely specified by the

function $C(f)$. Taking the moments of the generalized Kramers equation (1), it is shown in [10] that the time evolution of the density $\rho(\mathbf{r}, t)$ is governed by the generalized Smoluchowski equation

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left[\frac{1}{\xi} (\nabla p + \rho \nabla \Phi) \right]. \quad (5)$$

This equation was initially obtained in [12] from a special form of Kramers equation associated with the Fermi-Dirac entropy. The generalized Smoluchowski equation decreases the functional

$$F[\rho] = \int \rho \int_0^\rho \frac{p(\rho')}{\rho'^2} d\rho' d^3\mathbf{r} + \int \rho \Phi d^3\mathbf{r}, \quad (6)$$

which is the simplified form of free energy (2) obtained by using the isotropic distribution function (4) to express $F[f]$ as a functional of ρ [10]. Thus, $\dot{F} \leq 0$, as can be checked by a direct calculation. At equilibrium, $\dot{F} = 0$ and we obtain the condition of hydrostatic equilibrium

$$\nabla p + \rho \nabla \Phi = \mathbf{0}. \quad (7)$$

This relation also directly results from Eq. (3). The object of this paper is to present a formal derivation of the generalized Smoluchowski equation (5) from the generalized Kramers equation (1), using a Chapman-Enskog expansion.

III. CHAPMAN-ENSKOG EXPANSION OF THE GENERALIZED KRAMERS EQUATION

We write the generalized Kramers equation in the form

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \cdot \left\{ D \left[h(f) \frac{\partial f}{\partial \mathbf{v}} + \beta g(f) \mathbf{w} \right] \right\}, \quad (8)$$

where g and h are arbitrary (positive) functions [11]. We assume that our Brownian particles evolve in a background that is rotating and translating uniformly, so that $\mathbf{w} = \mathbf{v} - \mathbf{u}$ denotes the *relative* velocity of the particles with respect to the background flow with $\mathbf{u} = \boldsymbol{\Omega} \times \mathbf{r} + \mathbf{U}$. The “generalized entropy” associated with this equation can be written

$$S[f] = - \int C(f) d^3\mathbf{r} d^3\mathbf{v}, \quad (9)$$

where the function C is determined by the relation

$$C''(f) = \frac{h(f)}{g(f)}. \quad (10)$$

We consider the high friction limit $\xi = D\beta \rightarrow +\infty$, where $\beta = O(1)$. We rewrite Eq. (8) in the form

$$\frac{\partial f}{\partial t} + Lf = \frac{1}{\epsilon}Q(f), \quad (11)$$

where $\epsilon = D^{-1}$ is a small parameter, $L = \mathbf{v} \cdot \partial/\partial \mathbf{r} - \nabla \Phi \cdot \partial/\partial \mathbf{v}$ is the advective operator and $Q(f)$ is the collision operator

$$Q(f) = \frac{\partial}{\partial \mathbf{v}} \cdot \left[h(f) \frac{\partial f}{\partial \mathbf{v}} + \beta g(f) \mathbf{w} \right]. \quad (12)$$

Our aim is now to derive an equation for $\rho(\mathbf{r}, t)$ to second order with respect to ϵ . For that purpose, we follow a Chapman-Enskog procedure and employ a formalism in the spirit of [17]. More precisely, we look for a solution f to (11) of the form $f = f_0 + \epsilon f_1$ where f_0 is the equilibrium distribution function with the same density as f , that is, $Q(f_0) = 0$ and

$$\int f_0(\mathbf{r}, \mathbf{v}, t) d^3 \mathbf{v} = \rho(\mathbf{r}, t). \quad (13)$$

Substituting this expression in Eq. (11), we get

$$\left(\frac{\partial}{\partial t} + L \right) (f_0 + \epsilon f_1) = \frac{1}{\epsilon} Q(f_0 + \epsilon f_1), \quad (14)$$

$$= DQ(f_0)f_1 + O(\epsilon), \quad (15)$$

where $DQ(f_0)$ is the linearized collision operator given by

$$DQ(f_0)f_1 = \frac{\partial}{\partial \mathbf{v}} \cdot \left[h'(f_0)f_1 \frac{\partial f_0}{\partial \mathbf{v}} + h(f_0) \frac{\partial f_1}{\partial \mathbf{v}} + \beta g'(f_0)f_1 \mathbf{w} \right]. \quad (16)$$

It then follows from Eq. (15) that f_1 satisfies

$$DQ(f_0)f_1 = \left(\frac{\partial}{\partial t} + L \right) f_0 + O(\epsilon). \quad (17)$$

Next, integrating Eq. (8) over the velocities, we obtain

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\int f_0 \mathbf{v} d^3 \mathbf{v} \right) = -\epsilon \nabla \cdot \left(\int f_1 \mathbf{v} d^3 \mathbf{v} \right). \quad (18)$$

Our aim now is to express f_0 and f_1 in terms of ρ , so as to deduce from (18) a closed equation for ρ . First, we note that $Q(f_0) = 0$ is equivalent to the condition

$$C'(f_0) = -\beta \left[\frac{w^2}{2} + \lambda(\mathbf{r}, t) \right]. \quad (19)$$

This relation, supplemented by Eq. (13), completely specifies f_0 .

We next determine f_1 . For that purpose, in view of (17), we need to investigate the invertibility properties of $DQ(f_0)$. This is the subject of the following result.

Proposition III.1 *The operator $DQ(f_0)$ is a non-positive self-adjoint and closed operator on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, C''(f_0)d^3\mathbf{v})$. The null space \mathcal{N} of $DQ(f_0)$ is given by*

$$\mathcal{N} = \mathbb{R} \frac{1}{C''(f_0)} = \mathbb{R} \frac{g(f_0)}{h(f_0)} = \left\{ \lambda \frac{g(f_0)}{h(f_0)}, \lambda \in \mathbb{R} \right\}, \quad (20)$$

and the range of $DQ(f_0)$ is the orthogonal space to \mathcal{N} , denoted by \mathcal{N}^\perp .

Proof. We observe that the linearized collision operator takes the simpler form

$$DQ(f_0)F = \frac{\partial}{\partial \mathbf{v}} \cdot \left[g(f_0) \frac{\partial}{\partial \mathbf{v}} (C''(f_0)F) \right]. \quad (21)$$

Consequently, for $F \in \mathcal{H}$ and $G \in \mathcal{H}$, we have

$$\langle DQ(f_0)F, G \rangle_{\mathcal{H}} = - \int g(f_0) \frac{\partial}{\partial \mathbf{v}} (C''(f_0)F) \cdot \frac{\partial}{\partial \mathbf{v}} (C''(f_0)G) d^3\mathbf{v},$$

from which we readily deduce that $DQ(f_0)$ is non-positive and self-adjoint in \mathcal{H} . Also, its null space is given by Eq. (20). The closedness of $DQ(f_0)$ finally follows by classical arguments. \square

We are now in a position to determine f_1 . Let Π be the orthogonal projection in \mathcal{H} on \mathcal{N} and denote by $DQ(f_0)^{-1}$ the inverse of the restriction of $DQ(f_0)$ to \mathcal{N}^\perp . Then, applying successively $Id - \Pi$ and $DQ(f_0)^{-1}$ to (17), we obtain

$$f_1 = DQ(f_0)^{-1} (Id - \Pi) \left(\frac{\partial}{\partial t} + L \right) f_0 + O(\epsilon). \quad (22)$$

Now, noting that

$$C''(f_0) \frac{\partial f_0}{\partial \mathbf{v}} = -\beta \mathbf{w}, \quad (23)$$

$$C'''(f_0) \frac{\partial f_0}{\partial \mathbf{r}} = -\beta \nabla \lambda + \beta (\nabla \mathbf{u})^T \mathbf{w}, \quad (24)$$

and

$$C'''(f_0) \frac{\partial f_0}{\partial t} = -\beta \frac{\partial \lambda}{\partial t}, \quad (25)$$

we find that

$$\begin{aligned} \left(\frac{\partial}{\partial t} + L \right) f_0 &= -\frac{\beta}{C''(f_0)} \left(\frac{\partial \lambda}{\partial t} + \mathbf{u} \cdot \nabla \lambda - (\nabla \mathbf{u})^T \mathbf{w} \cdot \mathbf{w} \right) \\ &\quad - \frac{\beta}{C'''(f_0)} \left(\mathbf{w} \cdot \nabla \lambda - (\nabla \mathbf{u})^T \mathbf{w} \cdot \mathbf{u} - \mathbf{w} \cdot \nabla \Phi \right). \end{aligned} \quad (26)$$

In the above expressions, $\nabla \mathbf{u}$ designates the matrix $(\partial_i u_j)$ and T is the transposition, i.e. $Aw \cdot u = w \cdot A^T u$. Since $(\nabla \mathbf{u})$ is skew-adjoint, i.e. $(\nabla \mathbf{u})^T = -\nabla \mathbf{u}$, we have $(\nabla \mathbf{u}) \mathbf{w} \cdot \mathbf{w} = 0$ and thus,

$$(Id - \Pi) \left(\frac{\partial}{\partial t} + L \right) f_0 = -\frac{\beta}{C''(f_0)} \left(\nabla \lambda - (\nabla \mathbf{u}) \mathbf{u} - \nabla \Phi \right) \cdot \mathbf{w}. \quad (27)$$

Note that the application of $Id - \Pi$ on the first two terms of Eq. (26) yields zero since they belong to \mathcal{N} according to Proposition III.1. According to Eqs. (21), (22) and (27), the function f_1 is determined by the partial differential equation

$$C'''(f_0) \frac{\partial}{\partial \mathbf{v}} \cdot \left[g(f_0) \frac{\partial}{\partial \mathbf{v}} (C'''(f_0) f_1) \right] = -\beta \left(\nabla \lambda - (\nabla \mathbf{u}) \mathbf{u} - \nabla \Phi \right) \cdot \mathbf{w} + O(\epsilon). \quad (28)$$

Introducing the pressure

$$p = \frac{1}{3} \int f_0 w^2 d^3 \mathbf{v}, \quad (29)$$

and using Eqs. (23) and (24), we find after straightforward calculations that

$$\frac{1}{\rho} \nabla p = -\nabla \lambda. \quad (30)$$

Substituting the foregoing relation in Eq. (28), we obtain

$$C'''(f_0) \frac{\partial}{\partial \mathbf{v}} \cdot \left[g(f_0) \frac{\partial}{\partial \mathbf{v}} (C'''(f_0) f_1) \right] = \frac{\beta}{\rho} (\nabla p + \rho \nabla \Phi_{eff}) \cdot \mathbf{w} + O(\epsilon), \quad (31)$$

where

$$\Phi_{eff} = \Phi - \frac{u^2}{2} \quad (32)$$

is the *effective potential* accounting for inertial forces.

Since Eq. (31) is linear in f_1 , we look for solutions of the form

$$C'''(f_0) f_1 = -\frac{\beta}{\rho} \mathbf{R}(\mathbf{w}) \cdot (\nabla p + \rho \nabla \Phi_{eff}) + O(\epsilon). \quad (33)$$

Substituting this in Eq. (31), we find that \mathbf{R} must satisfy the equation

$$C'''(f_0) \frac{\partial}{\partial \mathbf{w}} \left[g(f_0) \frac{\partial R_i}{\partial \mathbf{w}} \right] = -w_i. \quad (34)$$

In other words, $DQ(f_0)(R_i/C'''(f_0)) = -w_i/C'''(f_0)$, which yields, together with (33),

$$-\epsilon \nabla \cdot \left(\int f_1 \mathbf{v} d^3 \mathbf{v} \right) = \nabla \cdot \left[\mathcal{X} (\nabla p + \rho \nabla \Phi_{eff}) \right], \quad (35)$$

where the matrix $\mathcal{X} = (\mathcal{X}_{i,j})_{1 \leq i,j \leq 3}$ is given by

$$\mathcal{X}_{i,j} = -\frac{\epsilon \beta}{\rho} \int R_i DQ(f_0) \left(\frac{R_j}{C'''(f_0)} \right) d^3 \mathbf{v} = -\frac{\epsilon \beta}{\rho} \left\langle \frac{R_i}{C'''(f_0)}, DQ(f_0) \left(\frac{R_j}{C'''(f_0)} \right) \right\rangle_{\mathcal{H}}. \quad (36)$$

In particular, \mathcal{X} is a non-negative matrix by Proposition III.1. Inserting the relation (35) in Eq. (18), we obtain the generalized Smoluchowski equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = \nabla \left[\mathcal{X}(\nabla p + \rho \nabla \Phi_{eff}) \right]. \quad (37)$$

Coming back to the equation (34) satisfied by \mathbf{R} , it can be written using Eq. (10) as

$$-h(f_0)\Delta_w \mathbf{R} + g'(f_0)\beta(\mathbf{w} \cdot \nabla_w)\mathbf{R} = \mathbf{w}. \quad (38)$$

This equation admits solutions of the form

$$\mathbf{R}(\mathbf{w}) = \frac{1}{\beta}\phi(x)\mathbf{w}, \quad (39)$$

where $x = \beta w^2/2$ and $\phi(x)$ is a solution of the differential equation

$$-h(f_0)(2x\phi'' + 5\phi') + g'(f_0)(\phi + 2x\phi') = 1. \quad (40)$$

We have thus established that the first order correction to the distribution function f_0 is given by

$$f_1 = -\frac{1}{C''(f_0)}\phi(x)\frac{1}{\rho}(\nabla p + \rho \nabla \Phi_{eff}) \cdot \mathbf{w}, \quad (41)$$

and the matrix \mathcal{X} defined in Eq. (36) by

$$\mathcal{X} = \chi I_3 \quad \text{with} \quad \chi = \frac{\epsilon}{3\rho} \int \frac{1}{C''(f_0)}\phi(x)w^2 d^3\mathbf{w}. \quad (42)$$

Equations (37), (40) and (42) formally solve the problem in the general case. We shall now consider simplified forms of the generalized Kramers equation, corresponding to specific expressions of g and h [11]. First, we impose $g(f) = f$ and $h(f) = fC''(f)$. In that case, the collision operator takes the form

$$Q(f) = \frac{\partial}{\partial \mathbf{v}} \cdot \left[fC''(f)\frac{\partial f}{\partial \mathbf{v}} + \beta f\mathbf{w} \right], \quad (43)$$

and the differential equation (40) becomes

$$-f_0C''(f_0)(2x\phi'' + 5\phi') + \phi + 2x\phi' = 1. \quad (44)$$

Clearly, $\phi = 1$ is a solution. Substituting this result in Eq. (42), we obtain

$$\begin{aligned} \chi &= \frac{\epsilon}{3\rho} \int \frac{1}{C''(f_0)}w^2 d^3\mathbf{w} = -\frac{\epsilon}{3\beta\rho} \int \frac{\partial f_0}{\partial \mathbf{w}} \cdot \mathbf{w} d^3\mathbf{w} \\ &= \frac{\epsilon}{\beta\rho} \int f_0 d^3\mathbf{w} = \frac{\epsilon}{\beta} = \frac{1}{D\beta} = \frac{1}{\xi}. \end{aligned} \quad (45)$$

Therefore, for this special class of generalized Kramers equations, the Smoluchowski equation is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = \nabla \cdot \left[\frac{1}{\xi} (\nabla p + \rho \nabla \Phi_{eff}) \right]. \quad (46)$$

This equation was previously derived in [10] by working out the moments of the Kramers equation. The present method, in addition of being more rigorous, also provides the first order correction to the distribution function. We have

$$f = f_0 - \epsilon \frac{1}{C'''(f_0)} \frac{1}{\rho} (\nabla p + \rho \nabla \Phi_{eff}) \cdot \mathbf{w} + O(\epsilon^2). \quad (47)$$

If we now impose $g(f) = 1/C''(f)$ and $h(f) = 1$, the collision operator takes the form

$$Q(f) = \frac{\partial}{\partial \mathbf{v}} \cdot \left[\frac{\partial f}{\partial \mathbf{v}} + \frac{\beta}{C''(f)} \mathbf{w} \right], \quad (48)$$

and the differential equation (40) becomes

$$-(2x\phi'' + 5\phi') - \frac{C'''(f_0)}{C''(f_0)^2} (\phi + 2x\phi') = 1. \quad (49)$$

There does not seem to be any simple solution to this equation. This implies that χ is probably not equal to $1/\xi$ in that case.

IV. A MOMENT METHOD FOR THE GENERALIZED KRAMERS EQUATION

In this section, we use a moment method to derive a variant of the Smoluchowski equation from the generalized Kramers equation. This approach was previously developed in [12] for the fermionic Kramers equation. We consider here a more general situation. We show in particular that the diffusion coefficient obtained by the moment method has a different expression from that obtained by the above Chapman-Enskog approach. To simplify the presentation, we consider the case $\mathbf{u} = 0$, that is

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \cdot \left\{ D \left[h(f) \frac{\partial f}{\partial \mathbf{v}} + \beta g(f) \mathbf{v} \right] \right\} = Q(f)(\mathbf{v}), \quad (50)$$

and seek equations satisfied by the macroscopic quantities associated with f (mass, impulse and internal energy) defined as

$$\begin{pmatrix} \rho \\ \rho \bar{\mathbf{u}} \\ \rho \bar{e} \end{pmatrix} = \int f(\mathbf{v}) \begin{pmatrix} 1 \\ \mathbf{v} \\ \frac{w^2}{2} \end{pmatrix} d^3 \mathbf{v}, \quad (51)$$

where $\mathbf{w} = \mathbf{v} - \bar{\mathbf{u}}(\mathbf{r}, t)$. Integrating Eq. (50) against 1, \mathbf{v} , and $v^2/2$, we classically get

$$\partial_t \rho + \nabla \cdot (\rho \bar{\mathbf{u}}) = 0,$$

$$\partial_t(\rho \bar{\mathbf{u}}) + \nabla \cdot (\rho \bar{\mathbf{u}} \bar{\mathbf{u}}) + \nabla \cdot \bar{P} + \rho \nabla \Phi = \int Q(f) \mathbf{v} d^3 \mathbf{v}, \quad (52)$$

$$\partial_t \bar{E} + \nabla \cdot [\bar{E} \bar{\mathbf{u}} + \bar{P} \bar{\mathbf{u}} + \bar{\mathbf{q}}] + \rho \bar{\mathbf{u}} \cdot \nabla \Phi = \frac{1}{2} \int Q(f) v^2 d^3 \mathbf{v},$$

where

$$\bar{E} = \rho \left(\frac{\bar{\mathbf{u}}^2}{2} + \bar{e} \right), \quad \bar{P} = \int f(\mathbf{v}) \mathbf{w} \mathbf{w} d^3 \mathbf{v}, \quad \text{and} \quad \bar{\mathbf{q}} = \int f(\mathbf{v}) \frac{w^2}{2} \mathbf{w} d^3 \mathbf{v}, \quad (53)$$

are the energy, the pressure tensor and the current of heat. By combining Eqs. (52-b) and (52-c), we can rewrite the equation for the energy as

$$\partial_t(\rho \bar{e}) + \nabla \cdot (\rho \bar{e} \bar{\mathbf{u}}) + \nabla \cdot \bar{\mathbf{q}} + \bar{P} \nabla \bar{\mathbf{u}} = \frac{1}{2} \int Q(f) w^2 d^3 \mathbf{v}. \quad (54)$$

System (52) is not closed in the sense that it involves unknown quantities $\bar{P}(f)$ and $\bar{\mathbf{q}}(f)$ that still depend on the distribution function f . It is well known that a maximum entropy principle can be used to specify f in terms of ρ , $\bar{\mathbf{u}}$ and \bar{e} leading to a closed set of equations for these quantities. Under some suitable hypotheses on the entropy functional $C(f)$, one can show that the distribution function f_* that maximizes the entropy density $-\int C(f) d^3 \mathbf{v}$ with the prescribed conditions (51) is given by

$$C'(f_*) = -\bar{\beta}(\mathbf{r}, t) \left(\frac{w^2}{2} + \lambda(\mathbf{r}, t) \right). \quad (55)$$

Then, we have

$$\bar{P}(f_*) = \bar{p} I_3, \quad \bar{p} = \frac{1}{3} \int f_*(\mathbf{v}) w^2 d^3 \mathbf{v}, \quad \text{and} \quad \bar{\mathbf{q}}(f_*) = \mathbf{0}. \quad (56)$$

We note in particular that $\bar{p} = \frac{2}{3} \rho \bar{e}$. On the other hand, a straightforward computation gives

$$\int Q(f_*) \mathbf{v} d^3 \mathbf{v} = -\frac{1}{\bar{\chi}} \rho \bar{\mathbf{u}}, \quad (57)$$

and

$$\int Q(f_*) \frac{w^2}{2} d^3 \mathbf{v} = \frac{1}{\epsilon} (\bar{\beta} - \beta) \int w^2 g(f_*) d^3 \mathbf{v}, \quad (58)$$

with

$$\bar{\chi} = \epsilon \frac{\rho}{\beta \int g(f_*) d^3 \mathbf{v}}. \quad (59)$$

We thus obtain the system of hydrodynamic equations

$$\begin{aligned}\partial_t \rho + \nabla \cdot (\rho \bar{\mathbf{u}}) &= 0, \\ \rho(\partial_t \bar{\mathbf{u}} + \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}}) &= -\nabla \bar{p} - \rho \nabla \Phi - \frac{1}{\bar{\chi}} \rho \bar{\mathbf{u}}, \\ \rho(\partial_t \bar{e} + \bar{\mathbf{u}} \cdot \nabla \bar{e}) + \bar{p} \nabla \cdot \bar{\mathbf{u}} &= \frac{1}{\epsilon} (\bar{\beta} - \beta) \int w^2 g(f_*) d^3 \mathbf{v}.\end{aligned}\tag{60}$$

This system of equations can be simplified in some particular situations. In the case $g(f) = f$ and $h(f) = f C''(f)$, we find that

$$\bar{\chi} = \frac{1}{\xi}, \quad \int w^2 g(f_*) d^3 \mathbf{v} = 2\rho \bar{e}.\tag{61}$$

In the case $h(f) = 1$ and $g(f) = 1/C''(f)$, we obtain

$$\bar{\chi} = \frac{\rho}{D\beta} \frac{1}{\int \frac{d^3 \mathbf{v}}{C''(f)}}, \quad \int w^2 g(f_*) d^3 \mathbf{v} = \frac{3\rho}{\beta},\tag{62}$$

where we have used a method similar to that of Eq. (45) to obtain the second equality. For the fermionic Kramers equation with $h(f) = 1$ and $g(f) = f(\eta_0 - f)$, we recover the equations obtained in [12]. Finally, for the classical Kramers equation with $h(f) = 1$ and $g(f) = f$, we have $p = \rho \bar{T}$ and $\bar{e} = \frac{3}{2} \bar{T}$ and the hydrodynamic equations

$$\begin{aligned}\partial_t \rho + \nabla \cdot (\rho \bar{\mathbf{u}}) &= 0, \\ \rho(\partial_t \bar{\mathbf{u}} + \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}}) &= -\nabla(\rho \bar{T}) - \rho \nabla \Phi - \xi \rho \bar{\mathbf{u}}, \\ \frac{3}{2}(\partial_t \bar{T} + \bar{\mathbf{u}} \cdot \nabla \bar{T}) + \bar{T} \nabla \cdot \bar{\mathbf{u}} &= 3\xi(T - \bar{T}).\end{aligned}\tag{63}$$

Returning to the general case (60) and considering the strong friction limit $\epsilon \rightarrow 0$, we get $\rho \bar{\mathbf{u}} = O(\epsilon)$, and the two last equations of (60) give

$$\rho \bar{\mathbf{u}} = \bar{\chi}(\nabla \bar{p} + \rho \nabla \Phi) + O(\epsilon^2), \quad \text{and} \quad \bar{\beta} = \beta + O(\epsilon).\tag{64}$$

Inserting this expression into the continuity equation (60-a), we get

$$\partial_t \bar{\rho} = \nabla \cdot [\bar{\chi}(\nabla \bar{p} + \rho \nabla \Phi)] + O(\epsilon^2).\tag{65}$$

This has a similar form as the model obtained in the above section, but with new formulae for the quantities \bar{p} and $\bar{\chi}$. Now observing that the difference between the distribution functions f_0 used in the Chapman-Enskog expansion and f_* (that maximizes the entropy) is of the order of ϵ , we get $\bar{p} = p + O(\epsilon)$. This means that, in formula (65), if one replaces \bar{p} by p , then one makes an error of the order ϵ^2 only. However, it is not clear whether $\chi = \bar{\chi} + O(\epsilon)$ or not. Explicit examples are difficult to construct because of the implicit dependence of χ in ρ , except for the case $g(f) = f$ where we have $\chi = \bar{\chi} = 1/\xi$.

V. LANGEVIN PARTICLES IN INTERACTION

A. A non-local Smoluchowski equation

The previous results remain valid when the force is related to the density by a relation of the form

$$\Phi(\mathbf{r}, t) = \int \theta(\mathbf{r}' - \mathbf{r}) \rho(\mathbf{r}', t) d^3 \mathbf{r}', \quad (66)$$

where $\theta(\mathbf{r}' - \mathbf{r})$ is a binary potential of interaction depending only on the absolute distance $|\mathbf{r}' - \mathbf{r}|$ between the particles. In that case, the potential energy reads

$$W = \frac{1}{2} \int \rho \Phi d^3 \mathbf{r}, \quad (67)$$

where the $1/2$ factor guarantees that the contribution of a particle is not counted twice. With this form of interaction, the generalized Smoluchowski equation (37) becomes non-local and takes the explicit form

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = \nabla \cdot \left\{ \chi \left[p'(\rho) \nabla \rho + \rho \int \frac{\partial \theta}{\partial \mathbf{r}}(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}', t) d^3 \mathbf{r}' + \rho \mathbf{\Omega} \times \mathbf{u} \right] \right\}, \quad (68)$$

where $\mathbf{u} = \mathbf{\Omega} \times \mathbf{r} + \mathbf{U}$. Equation (68) was introduced by Chavanis [10] (see an earlier version in [12]). It is valid for an arbitrary equation of state $p(\rho)$ and an arbitrary potential of interaction $\theta(\mathbf{r}' - \mathbf{r})$. When the potential of interaction is long-range and attractive, this equation exhibits a rich variety of behaviors associated with canonical phase transitions and blow up phenomena. Among long-range potentials, the gravitational potential plays an important role. In this context, Eq. (68) has been studied in [18] for an isothermal equation of state $p = \rho T$, in [19] for a polytropic equation of state $p = K \rho^\gamma$ and in [20] for a Fermi-Dirac equation of state. In the isothermal case, Eq. (68) describes a gas of Brownian particles in interaction (this is the canonical counterpart of a Hamiltonian system of particles in interaction). In the non-isothermal case, it describes Langevin particles in interaction displaying anomalous diffusion. A formal derivation of Eq. (68) is proposed in [21] starting from a generalized class of stochastic processes and using a mean-field approximation. Equations similar to non-local generalized Fokker-Planck equations also arise in the context of 2D turbulence and stellar dynamics [12, 22], and to describe the chemotaxis of bacterial populations [10, 23]. As explained in Refs. [10, 19], Eq. (68) can be used alternatively as a numerical algorithm to compute nonlinearly dynamically stable stationary solutions of the

Euler-Jeans equations in astrophysics. This can be particularly interesting in the case of rotating stars that are not spherically symmetric.

B. The Lyapunov functional

An additional interesting feature of the generalized Kramers and Smoluchowski equations when the potential Φ is given by Eq. (66) is the existence of a Lyapunov functional [10]. Indeed, let us write the generalized Kramers equation in the form

$$\frac{df}{dt} = \frac{\partial}{\partial \mathbf{v}} \cdot \left\{ D \left[h(f) \frac{\partial f}{\partial \mathbf{v}} + \beta g(f) \mathbf{w} \right] \right\} \equiv - \frac{\partial \mathbf{J}_f}{\partial \mathbf{v}}, \quad (69)$$

where $d/dt = \partial/\partial t + L$ is the material derivative and \mathbf{J}_f is the diffusion current. We introduce the functional

$$F[f] = E[f] - TS[f] - \mathbf{\Omega} \cdot \mathbf{L}[f] - \mathbf{U} \cdot \mathbf{P}[f], \quad (70)$$

where

$$E = \int f \frac{v^2}{2} d^3 \mathbf{r} d^3 \mathbf{v} + \frac{1}{2} \int \rho \Phi d^3 \mathbf{r}, \quad (71)$$

$$\mathbf{L} = \int f \mathbf{r} \times \mathbf{v} d^3 \mathbf{r} d^3 \mathbf{v}, \quad (72)$$

$$\mathbf{P} = \int f \mathbf{v} d^3 \mathbf{r} d^3 \mathbf{v}, \quad (73)$$

are the energy, the angular momentum and the impulse respectively. The “generalized entropy” S is defined by Eqs. (9) and (10). The functional (70) can be interpreted as a generalized free energy. Using the skew-adjointness of $\nabla \mathbf{u}$, it is straightforward to check that

$$\dot{F} = - \int \frac{J_f^2}{\xi g(f)} d^3 \mathbf{v} d^3 \mathbf{r} \leq 0, \quad (74)$$

so that $F[f]$ plays the role of a Lyapunov functional for the generalized Kramers equation. A stationary solution of Eq. (69), defined by $\partial f / \partial t = 0$, satisfies $\dot{F} = 0$. Hence, it cancels the diffusion current $\mathbf{J}_f = \mathbf{0}$ in virtue of Eq. (74). It must also cancel the advection term $Lf = 0$. Using these two conditions, we can show that f_{eq} is determined by

$$C'(f_{eq}) = -\beta \left(\frac{w^2}{2} + \Phi_{eff} \right) - \alpha. \quad (75)$$

Therefore, a stationary solution of the Kramers equation extremizes the free energy (70) at fixed mass, angular velocity, linear velocity and temperature. It can be shown furthermore that only *minima* of F are linearly stable via the Kramers equation [10].

The Lyapunov functional associated with the generalized Smoluchowski equation (68) can be obtained from Eq. (70) by replacing f by its leading term f_0 [10, 24]. Thus,

$$F[\rho] = E[f_0] - TS[f_0] - \mathbf{\Omega} \cdot \mathbf{L}[f_0] - \mathbf{U} \cdot \mathbf{P}[f_0]. \quad (76)$$

According to Eq. (19), f_0 can be written

$$f_0 = F \left[\beta \left(\frac{w^2}{2} + \lambda(\mathbf{r}, t) \right) \right], \quad (77)$$

where $F(x) = (C')^{-1}(-x)$. Using Eq. (77), the density $\rho = \int f_0 d^3\mathbf{v}$ and the pressure $p = \frac{1}{3} \int f_0 w^2 d^3\mathbf{v}$ can be put in the form

$$\rho = \frac{1}{\beta^{3/2}} G(\beta\lambda), \quad p = \frac{1}{\beta^{5/2}} H(\beta\lambda), \quad (78)$$

where

$$G(x) = 4\pi\sqrt{2} \int_0^{+\infty} F(x+t) t^{1/2} dt, \quad (79)$$

$$H(x) = \frac{8\pi\sqrt{2}}{3} \int_0^{+\infty} F(x+t) t^{3/2} dt. \quad (80)$$

We can now express E and S as functionals of ρ . We follow the derivation given in [24]. The energy (71) is easily expressed in terms of hydrodynamical variables as

$$E[f_0] = \frac{3}{2} \int p d^3\mathbf{r} + \frac{1}{2} \int \rho \Phi d^3\mathbf{r} + \frac{1}{2} \int \rho u^2 d^3\mathbf{r}. \quad (81)$$

On the other hand, the generalized entropy (9) can be written

$$S[f_0] = -\frac{4\pi\sqrt{2}}{\beta^{3/2}} \int d^3\mathbf{r} \int_0^{+\infty} C[F(t+\beta\lambda)] t^{1/2} dt. \quad (82)$$

Integrating by parts and using $C'[F(x)] = -x$, we find that

$$S[f_0] = -\frac{8\pi\sqrt{2}}{3\beta^{3/2}} \int d^3\mathbf{r} \int_0^{+\infty} F'(t+\beta\lambda) (t+\beta\lambda) t^{3/2} dt. \quad (83)$$

Integrating by parts one more time and using Eqs. (78), (79) and (80), we finally obtain

$$S[f_0] = \frac{5}{2}\beta \int p d^3\mathbf{r} + \beta \int \lambda \rho d^3\mathbf{r}. \quad (84)$$

The angular momentum and the linear impulse can be expressed as

$$\mathbf{L}[f_0] = \int \rho \mathbf{r} \times \mathbf{u} d^3 \mathbf{r}, \quad (85)$$

$$\mathbf{P}[f_0] = \int \rho \mathbf{u} d^3 \mathbf{r}. \quad (86)$$

Collecting all these results, the free energy (76) becomes

$$F[\rho] = - \int \rho \left(\lambda + \frac{p}{\rho} \right) d^3 \mathbf{r} + \frac{1}{2} \int \rho \Phi d^3 \mathbf{r} - \frac{1}{2} \int \rho u^2 d^3 \mathbf{r}. \quad (87)$$

Finally, using the relation $H'(x) = -G(x)$ obtained from Eqs. (79) and (80) by a simple integration by parts, it is easy to check that Eq. (78) implies

$$\lambda + \frac{p}{\rho} = - \int_0^\rho \frac{p(\rho')}{\rho'^2} d\rho'. \quad (88)$$

Hence, the free energy can be written more explicitly

$$F[\rho] = \int \rho \int_0^\rho \frac{p(\rho')}{\rho'^2} d\rho' d^3 \mathbf{r} + \frac{1}{2} \int \rho \Phi d^3 \mathbf{r} - \frac{1}{2} \int \rho u^2 d^3 \mathbf{r}. \quad (89)$$

This is the Lyapunov functional for the generalized Smoluchowski equation (68). Recalling that $\chi = \chi(\rho)$ is non-negative, a straightforward calculation shows that

$$\dot{F} = - \int \frac{1}{\rho} \chi(\rho) \mathbf{J}_\rho^2 d^3 \mathbf{r} \leq 0, \quad (90)$$

where \mathbf{J}_ρ is the diffusion current

$$\mathbf{J}_\rho = -(\nabla p + \rho \nabla \Phi_{eff}). \quad (91)$$

At equilibrium, $\mathbf{J}_\rho = \mathbf{0}$, and we obtain the condition of hydrostatic equilibrium in the rotating frame

$$\nabla p + \rho \nabla \Phi_{eff} = \mathbf{0}. \quad (92)$$

C. Generalized Cahn-Hilliard equation

The free energy functional (89) can be written explicitly

$$F[\rho] = \int \Gamma(\rho) d^3 \mathbf{r} + \frac{1}{2} \int \int \rho(\mathbf{r}) \theta(\mathbf{r}' - \mathbf{r}) \rho(\mathbf{r}') d^3 \mathbf{r} d^3 \mathbf{r}' - \frac{1}{2} \int \rho u^2 d^3 \mathbf{r}, \quad (93)$$

where we have defined

$$\Gamma(\rho) = \rho \int_0^\rho \frac{p(\rho')}{\rho'^2} d\rho'. \quad (94)$$

Noting that

$$\rho\Gamma''(\rho) = p'(\rho), \quad (95)$$

and taking the functional derivative of $F[\rho]$, we observe that the generalized Smoluchowski equation (68) can be written

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = \nabla \cdot \left(\chi \rho \nabla \frac{\delta F}{\delta \rho} \right). \quad (96)$$

If we now consider the case of short-range interactions, it is possible to expand the potential

$$\Phi(\mathbf{r}, t) = \int \theta(\mathbf{r}') \rho(\mathbf{r} + \mathbf{r}') d^3 \mathbf{r}' \quad (97)$$

in Taylor series for $\mathbf{r}' \rightarrow \mathbf{0}$. Introducing the notations

$$a = 4\pi \int_0^{+\infty} \theta(x) x^2 dx \quad \text{and} \quad b = \frac{4\pi}{3} \int_0^{+\infty} \theta(x) x^4 dx, \quad (98)$$

we obtain to second order

$$\Phi(\mathbf{r}, t) = a\rho(\mathbf{r}, t) + \frac{b}{2} \Delta \rho(\mathbf{r}, t). \quad (99)$$

In that limit, the free energy takes the form

$$F[\rho] = -\frac{b}{2} \int \left\{ \frac{(\nabla \rho)^2}{2} + V(\rho) \right\} d^3 \mathbf{r}, \quad (100)$$

where we have set $V(\rho) = -2\Gamma(\rho)/b - (a/b)\rho^2 + (1/b)\rho u^2$. This is the usual expression of the Landau free energy. In general b is negative so we have to minimize the functional integral. On the other hand, in this case of short-range interactions, the conservative equation (96) becomes

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = \nabla \cdot \left\{ \frac{b\chi}{2} \rho \nabla (\Delta \rho - V'(\rho)) \right\}. \quad (101)$$

This is the Cahn-Hilliard equation which has been extensively studied in the theory of phase ordering kinetics. Its stationary solutions describe “domain walls”. We can view therefore Eq. (68) as a generalization of the Cahn-Hilliard equation to the case of long-range interactions and arbitrary free energy functionals. This is why its physical and mathematical richness is so important.

VI. CONCLUSION

In this paper we have derived the Smoluchowski equation from the Kramers equation by using a formal Chapman-Enskog expansion. We have shown furthermore that this derivation could be generalized to a larger class of Kramers equations where the diffusion coefficient and the friction term depend on the distribution function. The resulting Smoluchowski equation takes a nice form where the usual term $T\rho$ is replaced by a pressure $p(\rho)$, as previously noted in [10, 12]. When the potential $\Phi(\mathbf{r}, t)$ is induced by the density of particles themselves, the generalized Smoluchowski equation becomes non-local and can generate a rich variety of phase transitions as discussed in [10], and more specifically in [18, 19, 20]. In the limit of short-range interactions, it reduces to the Cahn-Hilliard equation widely studied in the context of phase ordering kinetics. It is interesting to note that these different equations are closely related to each other and that they are associated with an *effective* generalized thermodynamical formalism [10, 11]. An interest of this formalism is to unify different types of approaches. For example, it encompasses the case of quantum particles with exclusion or inclusion principle. On the other hand, our approach extends the nonlinear Fokker-Planck equations [13] associated with the Tsallis entropy. Indeed, the Tsallis entropy does not play any special role in our formalism and most of the properties of the nonlinear Fokker-Planck equation can be generalized to a wider class of entropy functionals [10]. The Tsallis entropy forms, however, an important class of functionals associated with polytropic distributions and power-laws. These distributions generate a natural form of self-confinement that can be of interest in nonextensive systems. However, it was our interest here to show that a more general formalism could be developed consistently. A notion of “generalized thermodynamics”, with a different presentation and a different motivation, has been developed independently by Kaniadakis [8], Frank [9] and Naudts [25].

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APPENDIX A: BGK OPERATOR

In this Appendix, we derive the generalized Smoluchowski equation from a simplified form of kinetic equations. Specifically, we consider a BGK collision operator so that

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = -\frac{1}{\epsilon} \frac{f - f_0}{\tau}, \quad (\text{A1})$$

where $\epsilon\tau = \epsilon\tau(\rho)$ is a “typical collision” time and f_0 is defined by

$$C'(f_0) = -\beta \left(\frac{w^2}{2} + \lambda(\mathbf{r}, t) \right), \quad (\text{A2})$$

with $\int f_0 d^3\mathbf{v} = \rho$, $\mathbf{u} = \mathbf{U} + \boldsymbol{\Omega} \times \mathbf{r}$ and $\mathbf{w} = \mathbf{v} - \mathbf{u}$. We note that contrary to the usual BGK operator, the velocity distribution f_0 is *isotropic* (in the rotating frame). This is consistent with the properties of the generalized Kramers equation (8). We consider the limit $\epsilon \rightarrow 0$. Writing $f_1 = (f - f_0)/\epsilon$, we get to leading order

$$f_1 = -\tau \left(\frac{\partial}{\partial t} + L \right) f_0 + O(\epsilon). \quad (\text{A3})$$

Integrating Eq. (A1) over the velocities, we obtain

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\int f \mathbf{v} d^3\mathbf{v} \right) = 0, \quad (\text{A4})$$

or, using Eq. (A2),

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = -\epsilon \nabla \cdot \left(\int f_1 \mathbf{v} d^3\mathbf{v} \right). \quad (\text{A5})$$

Now,

$$\int f_1 \mathbf{v} d^3\mathbf{v} = -\tau \int \mathbf{v} \left(\frac{\partial}{\partial t} + L \right) f_0 d^3\mathbf{v} + O(\epsilon). \quad (\text{A6})$$

The first term in the foregoing equation can be written

$$\int \mathbf{v} \frac{\partial f_0}{\partial t} d^3\mathbf{v} = \frac{\partial}{\partial t} \int f_0 \mathbf{v} d^3\mathbf{v} = \mathbf{u} \frac{\partial \rho}{\partial t}. \quad (\text{A7})$$

To leading order, Eq. (A5) reduces to

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = O(\epsilon). \quad (\text{A8})$$

Therefore,

$$\int \mathbf{v} \frac{\partial f_0}{\partial t} d^3\mathbf{v} = -\mathbf{u} \nabla \cdot (\rho \mathbf{u}) + O(\epsilon). \quad (\text{A9})$$

The second term in Eq. (A6) can be written

$$\int v_i L f_0 d^3 \mathbf{v} = \int v_i \left(v_j \frac{\partial f_0}{\partial x_j} - \frac{\partial \Phi}{\partial x_j} \frac{\partial f_0}{\partial v_j} \right) d^3 \mathbf{v} = \frac{\partial}{\partial x_j} \int f_0 v_i v_j d^3 \mathbf{v} + \rho \frac{\partial \Phi}{\partial x_i}. \quad (\text{A10})$$

Now,

$$\int f_0 v_i v_j d^3 \mathbf{v} = p \delta_{ij} + \rho u_i u_j, \quad (\text{A11})$$

where $p = \frac{1}{3} \int f_0 w^2 d^3 \mathbf{v}$ is the pressure. After straightforward calculations, we find that

$$\frac{\partial}{\partial x_j} \int f_0 v_i v_j d^3 \mathbf{v} = \frac{\partial p}{\partial x_i} + u_i \nabla \cdot (\rho \mathbf{u}) + \rho (\boldsymbol{\Omega} \times \mathbf{u})_i. \quad (\text{A12})$$

Combining the foregoing results, we obtain

$$\int f_1 \mathbf{v} d^3 \mathbf{v} = -\tau (\nabla p + \rho \nabla \Phi + \rho \boldsymbol{\Omega} \times \mathbf{u}) + O(\epsilon). \quad (\text{A13})$$

Substituting this in Eq. (A5) and omitting terms of order ϵ^2 , we obtain the generalized Smoluchowski equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = \epsilon \nabla \cdot \left[\tau (\nabla p + \rho \nabla \Phi_{eff}) \right]. \quad (\text{A14})$$

APPENDIX B: TIME DEPENDENT LAGRANGE MULTIPLIERS

In this Appendix, we consider the generalized Kramers equation

$$\frac{\partial f}{\partial t} + Lf = \frac{\partial}{\partial \mathbf{v}} \cdot \left\{ D \left[h(f) \frac{\partial f}{\partial \mathbf{v}} + \beta(t) g(f) (\mathbf{v} - \boldsymbol{\Omega}(t) \times \mathbf{r} - \mathbf{U}(t)) \right] \right\}, \quad (\text{B1})$$

where the parameters β , $\boldsymbol{\Omega}$ and \mathbf{U} depend on time so as to satisfy at each instant t , the conservation of energy, angular momentum and impulse, and the potential Φ is given by Eq. (66). Their evolution is obtained by substituting Eq. (B1) in the constraints $\dot{E} = \dot{\mathbf{L}} = \dot{\mathbf{P}} = \mathbf{0}$. It can be shown that Eq. (B1) increases the generalized entropy (9) at fixed mass, energy, angular momentum and impulse (H-theorem). This corresponds to a microcanonical situation. Equation (B1) can be obtained from a variational principle called Maximum Entropy Production Principle (MEPP). In this context, β , $\boldsymbol{\Omega}$ and \mathbf{U} appear as time dependant Lagrange multipliers associated with the integral constraints. This type of equations were proposed in [12] as a small-scale parametrization of the gravitational Vlasov-Poisson system in the context of the theory of violent relaxation. As explained in [10], these relaxation equations can also be used as numerical algorithms to compute nonlinearly dynamically stable stationary solutions of the Vlasov-Poisson system. This would be particularly interesting in the

case of rotating stellar systems that are non spherically symmetric. They can have similar applications in other domains of physics with different types of interaction.

Considering the limit $\xi \rightarrow +\infty$, the Chapman-Enskog method of Sec. III can be implemented with only slight modifications. Due to the time dependence of the Lagrange multipliers, Eq. (25) is replaced by

$$C''(f_0) \frac{\partial f_0}{\partial t} = -\beta \frac{\partial \lambda}{\partial t} + \beta \mathbf{w} \cdot \dot{\mathbf{u}} - \dot{\beta} \left(\frac{w^2}{2} + \lambda \right), \quad (\text{B2})$$

where f_0 is still given by Eq. (19) and $\mathbf{u} = \mathbf{U} + \boldsymbol{\Omega} \times \mathbf{r}$. Consequently, introducing the notation

$$\nu = \int \frac{1}{C''(f_0)} d^3 \mathbf{v}, \quad (\text{B3})$$

and using the identity

$$\int \frac{1}{C''(f_0)} \frac{w^2}{2} d^3 \mathbf{v} = -\frac{1}{2\beta} \int \frac{\partial f_0}{\partial \mathbf{w}} \mathbf{w} d^3 \mathbf{v} = \frac{3\rho}{2\beta}, \quad (\text{B4})$$

resulting from Eq. (23), Eq. (27) becomes

$$(Id - \Pi) \left(\frac{\partial}{\partial t} + L \right) f_0 = -\frac{\beta}{C''(f_0)} \left(\nabla \lambda - \dot{\mathbf{u}} - (\nabla \mathbf{u}) \mathbf{u} - \nabla \Phi \right) \cdot \mathbf{w} - \frac{\dot{\beta}}{C''(f_0)} \left(\frac{w^2}{2} - \frac{3\rho}{2\nu\beta} \right). \quad (\text{B5})$$

Using Eqs. (B5), we find that Eq. (28) is replaced by

$$\begin{aligned} f_1 = & -\beta DQ(f_0)^{-1} \left(\frac{1}{C''(f_0)} (\nabla \lambda - \dot{\mathbf{u}} - (\nabla \mathbf{u}) \mathbf{u} - \nabla \Phi) \cdot \mathbf{w} \right) \\ & - \dot{\beta} DQ(f_0)^{-1} \left(\frac{1}{C''(f_0)} \left(\frac{w^2}{2} - \frac{3\rho}{2\nu\beta} \right) \right). \end{aligned} \quad (\text{B6})$$

Since $f_0(\mathbf{w})$ is spherically symmetric, only the terms proportional to \mathbf{w} in the right-hand side of Eq. (B6) will give a non-zero contribution in Eq. (18). Therefore, Eq. (37) remains valid with the substitution $\nabla \Phi \rightarrow \nabla \Phi + \dot{\mathbf{u}}$. We thus obtain the generalized Smoluchowski equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = \nabla \cdot \left[\chi (\nabla p + \rho \nabla \Phi_{eff} + \rho \dot{\mathbf{u}}) \right]. \quad (\text{B7})$$

Note that the last term was forgotten in [12]. The time dependent Lagrange multipliers $\beta(t)$, $\boldsymbol{\Omega}(t)$ and $\mathbf{U}(t)$ are determined by replacing f by its leading term f_0 in the conservation of energy, angular momentum and impulse. Recalling (81), (85) and (86), we obtain

$$\begin{aligned} \frac{3}{2} \int p(\rho) d^3 \mathbf{r} + \frac{1}{2} \int \rho \Phi d^3 \mathbf{r} + \frac{1}{2} \int \rho u^2 d^3 \mathbf{r} &= \text{const.}, \\ \int \rho \mathbf{r} \times \mathbf{u} d^3 \mathbf{r} &= \text{const.}, \\ \int \rho \mathbf{u} d^3 \mathbf{r} &= \text{const.}, \end{aligned}$$

where $p(\rho)$ is given by (78), (79) and (80). This type of equations with time dependant temperature has been studied in [18, 19, 26, 27, 28].

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